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On a Class of Functionals Whose Local Minima are Global

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Abstract. In this note we introduce a suitable class of functionals, including the class of integral functionals, and prove that any (strict) local minimum of a functional of this class, defined on a decomposable space, is a (strict) global minimum. So, the recent result obtained by Giner in [1] is specified and extended.

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An important topic in optimization theory is to see whether a given functional has the property of its local minima being global (see, for instance, [3] and the references therein).

Recently, in [1], Giner provided a remarkable contribution by proving that integral functionals on decomposable subsets of measurable functions, when endowed with a suitable topology, have the considered property.

The aim of the present note is simply to point out that the above result can be extended to a much broader class of functionals (Theorem 1) and specified when strict local minima are considered (Theorem 2).

Throughout the sequel, $(T, \mathcal{F}, \mu)$ is a σ-finite non-atomic measure space ($\mu(T) > 0$), E is a real separable Banach space, and X is a non-empty set of equivalence classes of measurable functions from T into E, two functions being equivalent if, out of a set of measure zero, they are equal. We assume that X is decomposable. This means that if $A \in \mathcal{F}$ and u, $v \in X$, then $1_A u + 1_{T\setminus A} v \in X$, where 1_A is the characteristic function of A. We will consider X endowed with a given topology τ such that if $\{A_n\}$ is a sequence in $\mathcal{F}$, with $\lim_{n\to\infty} \mu(A_n) = 0$, and u, $v \in X$, then the sequence $\{1_{A_n} u + 1_{T\setminus A_n} v\}$ τ-converges to v. We also denote by M the set of equivalence classes of measurable functions from T into $\overline{\mathbb{R}}$ (the equivalence relation being as above). Let us now introduce the classes of functionals we will deal with.

DEFINITION 1. A functional $J : M \longrightarrow \overline{\mathbb{R}}$ is said to be increasing if, for every α, $\beta \in M$ such that $\alpha(t) \le \beta(t)$ μ-a.e. in T, one has $J(\alpha) \le J(\beta)$.

DEFINITION 2. A functional $J : M \longrightarrow \overline{\mathbb{R}}$ is said to be strictly increasing if it is increasing and, for every α, $\beta \in M$ such that $\alpha(t) \leq \beta(t)$ μ-a.e. in T, $\mu(\{t \in T : \alpha(t) < \beta(t)\}) > 0$ and $J(\beta) \in \mathbb{R}$, one has $J(\alpha) < J(\beta)$.

Finally, we fix a function $f : T \times E \longrightarrow \overline{\mathbb{R}}$ such that, for each $u \in X$, the function $t \longrightarrow f(t, u(t))$ belongs to M. For any $J : M \longrightarrow \overline{\mathbb{R}}$, we define the functional J_f on X by putting

$$J_f(u) = J(f(\cdot, u(\cdot)))$$

for every $u \in X$.

Moreover, we recall that $u \in X$ is said to be a local minimum [strict local minimum] of J_f if there is a $\tau-$neighborhood U of u such that for every $v \in U$, $v \neq u$, one has $J_f(u) \leq J_f(v)$ [$J_f(u) < J_f(v)$] and that $u \in X$ is said to be a global minimum [strict global minimum] of J_f if for every $v \in X$, $v \neq u$, one has $J_f(u) \leq J_f(v)$ [$J_f(u) < J_f(v)$].

The following lemma plays a fundamental role in the proof of our main theorems and it can be proved using classical results of measure theory (see, for instance, [2]). For the reader's convenience, we give an explicit proof here.

LEMMA 1. *Let $A \subseteq T$ be a measurable set such that $\mu(A) > 0$. Then, there exists a sequence $\{A_n\}$ of measurable subsets of A such that $\mu(A_n) > 0$ for every $n \in \mathbb{N}$ and* $\lim_{n\to\infty} \mu(A_n) = 0$.

Proof. We can suppose $\mu(A) < +\infty$. Since μ is a non-atomic measure there is a measurable set $B \subset A$ such that $0 < \mu(B) < \mu(A)$. We denote by A_1 either B or its complement in A so that $0 < \mu(A_1) < \frac{1}{2}\mu(A)$. Therefore, applying an iterative procedure, the conclusion is obtained. □

Now, we can establish the following

THEOREM 1. *Let $J : M \longrightarrow \overline{\mathbb{R}}$ be a strictly increasing functional. Then, any local minimum u of J_f in X, such that $J_f(u) \in \mathbb{R}$ is a global minimum. Moreover, for every $v \in X$, one has $f(t, u(t)) \leq f(t, v(t))$ μ-a.e. in T .*

Proof. Let u be a local minimum of J_f in X such that $J_f(u) \in \mathbb{R}$. Arguing by contradiction, assume that there is $w \in X$ such that $J_f(w) < J_f(u)$. Put

$$A = \{t \in T : f(t, w(t)) < f(t, u(t))\}.$$

Of course $A \in \mathcal{F}$ and $\mu(A) > 0$, since J is increasing. Taking into account Lemma 1, we can find a sequence $\{A_n\}$ in $\mathcal{F}$ so that $A_n \subseteq A$, $\mu(A_n) > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(A_n) = 0$. Now, for each $n \in \mathbb{N}$, put

$$u_n = 1_{A_n} w + 1_{T \setminus A_n} u.$$

Since X is decomposable, $u_n \in X$. Moreover, we have $f(t, u_n(t)) \leq f(t, u(t))$ for all $t \in T$ and $\mu(\{t \in T : f(t, u_n(t)) < f(t, u(t))\}) > 0$. Consequently, as J is strictly increasing, we have $J_f(u_n) < J_f(u)$ for all $n \in \mathbb{N}$. But, $\{u_n\}$ τ-converges to u, and so u cannot be a local minimum for J_f, a contradiction.

Let us now prove the other statement of the theorem. Arguing by contradiction again, assume that there is $v \in X$ such that $\mu(\{t \in T : f(t, u(t)) > f(t, v(t))\}) > 0$. If we put $V = \{t \in T : f(t, u(t)) > f(t, v(t))\}$ and $z = 1_V v + 1_{T \setminus V} u$, we obtain $f(t, z(t)) \leq f(t, u(t))$ for all $t \in T$ and $\mu(\{t \in T : f(t, z(t)) < f(t, u(t))\}) > 0$. So, J being strictly increasing, one has $J_f(z) < J_f(u)$ and, since u is a global minimum, this is a contradiction. □

Our second result is as follows:

THEOREM 2. *Let $J : M \longrightarrow \overline{\mathbb{R}}$ be an increasing functional. Then, any strict local minimum u of J_f in X is a strict global minimum. Moreover, for every $v \in X \setminus \{u\}$, one has $f(t, u(t)) \leq f(t, v(t))$ μ-a.e. in T and $\mu(\{t \in T : f(t, u(t)) < f(t, v(t))\}) > 0$.*

Proof. Let u be a strict local minimum of J_f in X. Arguing by contradiction, assume that there is $w \in X \setminus \{u\}$ such that $J_f(w) \leq J_f(u)$.

Put

$$A = \{t \in T : f(t, w(t)) < f(t, u(t))\}.$$

If $\mu(A) > 0$, choosing $\{A_n\}$ and $\{u_n\}$ exactly as in the proof of Theorem 1, we have $\tau - \lim_{n\to\infty} u_n = u$, $u_n \neq u$ and $J_f(u_n) \leq J_f(u)$, against the fact that u is a strict local minimum of J_f.

If $\mu(A) = 0$, then we have $f(t, w(t)) \geq f(t, u(t))$ μ-a.e. in T.

Put

$$B = \{t \in T : w(t) \neq u(t)\}.$$

Clearly, $B \in \mathcal{F}$ and $\mu(B) > 0$. Choose a sequence $\{B_n\}$ in $\mathcal{F}$ with $B_n \subseteq B$, $\mu(B_n) > 0$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \mu(B_n) = 0$. Define $w_n : T \longrightarrow E$ by

$$w_n = 1_{B_n} w + 1_{T \setminus B_n} u.$$

Observe that $\tau - \lim_{n\to\infty} w_n = u$, and $w_n \neq u$, $J_f(w_n) \leq J_f(w) \leq J_f(u)$ for all $n \in \mathbb{N}$, which yields the desired contradiction.

Now, as in Theorem 1 it is possible to prove that for every $v \in X$, $v \neq u$ one has $f(t, u(t)) \leq f(t, v(t))$ μ-a.e. in T. Therefore, taking into account that $J_f(u) < J_f(v)$, one also has $\mu(\{t \in T : f(t, u(t)) < f(t, v(t))\}) > 0$. □

REMARK 1. The situation considered in [1] is as follows: f is $\mathcal{F} \otimes \mathcal{B}(E)$-measurable ($\mathcal{B}(E)$ being the Borel family of E) and J is the functional defined by

$$\int_T^* \alpha(t)d\mu = inf\{\int_T \beta(t)d\mu : \beta \in L^1(T), \quad \alpha(t) \leq \beta(t) \quad \mu - a.e. \text{ in } T\},$$

for every $\alpha \in M$. Of course J is strictly increasing. So, Theorems 1 and 2 extend and specify the result of [1].

REMARK 2. The functional defined on M as follows

$$J(\alpha) = \operatorname*{ess\,inf}_{t \in T} \alpha(t) \qquad [\text{ or } \quad J(\alpha) = \operatorname*{ess\,sup}_{t \in T} \alpha(t)]$$

is another easy example of an increasing functional to which Theorem 2 can be applied.

REMARK 3. If J is a simple increasing functional, a local minimum of J_f in X need not be a global minimum. Let us give an example. To this end, choose: $T = [0, 1]$ with the Lebesgue measure structure; $X = L^1([0, 1])$ with the usual norm; $E = \mathbb{R}$ and $J : M \longrightarrow \mathbb{R}$ the functional defined as follows:

$$J(\alpha) = \begin{cases} 1 & \text{if } \alpha(t) > 0 \text{ for a.e. } t \in [0,1] \\ -1 & \text{if } \alpha(t) < -1 \text{ for a.e. } t \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Moreover, let $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(t, z) = z$ for every $(t, z) \in [0, 1] \times \mathbb{R}$.

Clearly, J is an increasing functional, but it is not a strictly increasing functional; and $J_f(u) = J(u)$ for every $u \in X$. Now, we show that the function

$$\overline{u}(t) = \begin{cases} 1/2 & \text{if } t \in [0, 1/2] \\ 0 & \text{if } t \in]1/2, 1] \end{cases}$$

is a local minimum of J_f in X. To this end, we observe that if $v \in B(\overline{u}, 1/2) = \{u \in X : \| \overline{u} - u\|_1 < 1/2\}$, then $\| v\|_1 < 3/4$; hence $J_f(v) \neq -1$ ($J_f(v) = -1$ implies $\| v\|_1 > 1$). So, taking into account that $J_f(\overline{u}) = 0$, $\overline{u}$ is a local minimum of J_f in X, but it is not a global minimum (take, for instance, $w(t) = -2$ for every $t \in T$).

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